PHYS 705: Classical Mechanics

Reminder:

- HW #6 is out now so that you can start on it early
- Columbus Day Holiday, class will meet on Tuesday (Oct 12)
- Be careful with online resources

HW #4 2.16: Conserved Physical Quantities

There is a very strong link between Symmetry and Conservation Theorems:

Symmetry ⟺ conserved quantity

cyclic coordinate $\equiv$ one that doesn't appear in L, *i.e.*, $\dfrac{\partial L}{\partial q_j} = 0$

generalized momentum $p_j = \dfrac{\partial L}{\partial \dot{q}_j} = const$

p_j is conserved !

Energy Conservation and Time Invariance

Defining $h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$

1. If L is time invariance (t is cyclic in L), h is conserved.

$$\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0$$

h is constant in time!

2. Additionally, if

$U = U(q_i)$ (U does not dep on $\dot{q}_i$) AND

$\frac{\partial \mathbf{r}_i}{\partial t} = 0$ (the coord trans does not dep on time explicitly)

$\Rightarrow$ $h = E$ (total energy)!

Energy Conservation and Time Invariance

Conservation of h (Jacobi Integral):

shown in class (check class note)

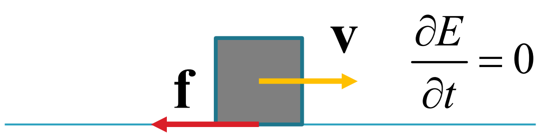
$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad h \text{ is conserved!}$$

But, in general $\frac{\partial h}{\partial t} = 0 \quad \not\Rightarrow \quad h$ is conserved

Or, vice versa, $\frac{\partial h}{\partial t} \neq 0 \quad \not\Rightarrow \quad h$ is NOT conserved

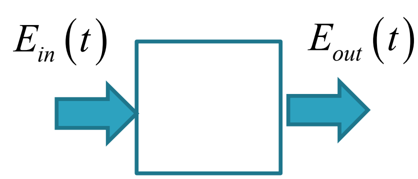
recall... $\left[\frac{\partial h}{\partial t} \neq \frac{dh}{dt} \right]$

$$E = \frac{1}{2}mv^2 - W_f$$



$$\frac{\partial E}{\partial t} = 0 \quad but \quad E \downarrow$$

Slowing down with friction



$$\frac{\partial E_{in}}{\partial t} \neq 0 \;\&\; \frac{\partial E_{out}}{\partial t} \neq 0$$

$$but \quad E_{tot} = const$$

A energy balanced steady-state system

HW #4 2.16 (Conservation of E and h)

$$L = e^{\gamma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

What are the conserved quantities?

$$\ddot{q} + \gamma\dot{q} + \frac{k}{m}q = 0$$

What physical system does this ODE describes?

Transform the system with a point transform: $s = e^{\gamma t/2}q$

$$L = \left(\frac{m\gamma^2}{8} - \frac{k}{2}\right)s^2 - \frac{m\gamma}{2}s\dot{s} + \frac{m}{2}\dot{s}^2$$

$$\ddot{s} + \left(\frac{k}{m} - \frac{\gamma^2}{4}\right)s = 0$$

Now, what are the conserved quantities?

HW #4 2.16 (Conservation of E and h)

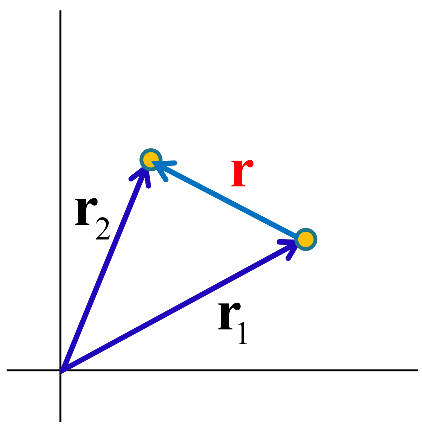
Also, k and γ are fixed parameters ("constants") in the problem BUT they are not **Constants of Motion** nor conserved quantities !

Two-Body Central Force Problem

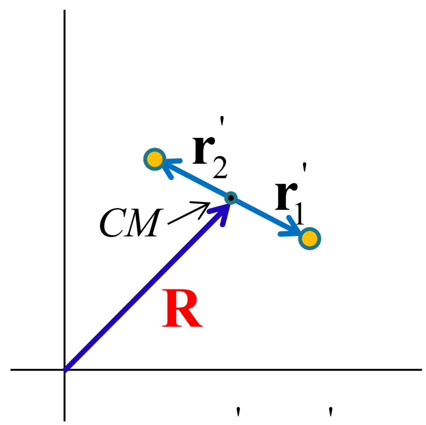
Set up of the general problem: 2 masses interact via forces directed along the line that connects them (central force): strong form of 3rd law

First Step: Central force problems can be reduced to an effective 1-body problem:

Change to generalized coordinates: CM position **R** and Relative position **r**,



$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$



$$\mathbf{r} = \mathbf{r}_2' - \mathbf{r}_1' \qquad \mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1'$$

$$\mathbf{r}_2 = \mathbf{R} + \mathbf{r}_2'$$

Instead of using $\mathbf{r}_1$ & $\mathbf{r}_2$, use **R & r** (CM & relative position)

Two-Body Central Force Problem

So,

$$L = T - U = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - U(\mathbf{r})$$

R does not appear in *L* (cyclic) so that EL equation for **R** will only give:

$$M\dot{\mathbf{R}} = const$$

Pick an inertial ref. frame (CM frame) in which CM is *not* moving $\dot{\mathbf{R}} = 0$ and we can ignore the 1st term in *L*.

The result is then:

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r}), \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Same as a **single particle** with mass μ moving in $U(\mathbf{r})$.

Two-Body Central Force Problem

→ $U(\mathbf{r}) = U(r)$ is central, the problem will be cyclic in an angular variable about any fixed axis about O → so that angular momentum is conserved.

→ **L** will points in a constant direction fixed by initial condition →motion has to be planar.

We can then analyze the problem entirely on a plane $\perp$ to L using polar coord.

And, the Lagrangian is given by:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

(where $m=\mu$ is the reduced mass)

Two-Body Central Force Problem

Summary: We get 2 EOMs and 2 integrals of motion (l, E) for this problem.

$$mr^2\dot{\theta} = l$$

$$m\ddot{r} = -\frac{dU}{dr} + \frac{l^2}{mr^3}$$

$$\frac{m}{2} r^2 \dot{\theta}^2$$

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + U(r)$$

Note: The E equation effectively is the 1st integral of the $\ddot{r}$ equation. We can rewrite it explicitly as,

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{l^2}{2mr^2}\right)}$$

Graphical Analysis of Central Force Problem

Using the concept of an **effective potential**, one can get an useful *qualitative* understanding of the problem without actually integrating!

Let consider the *r* equation:

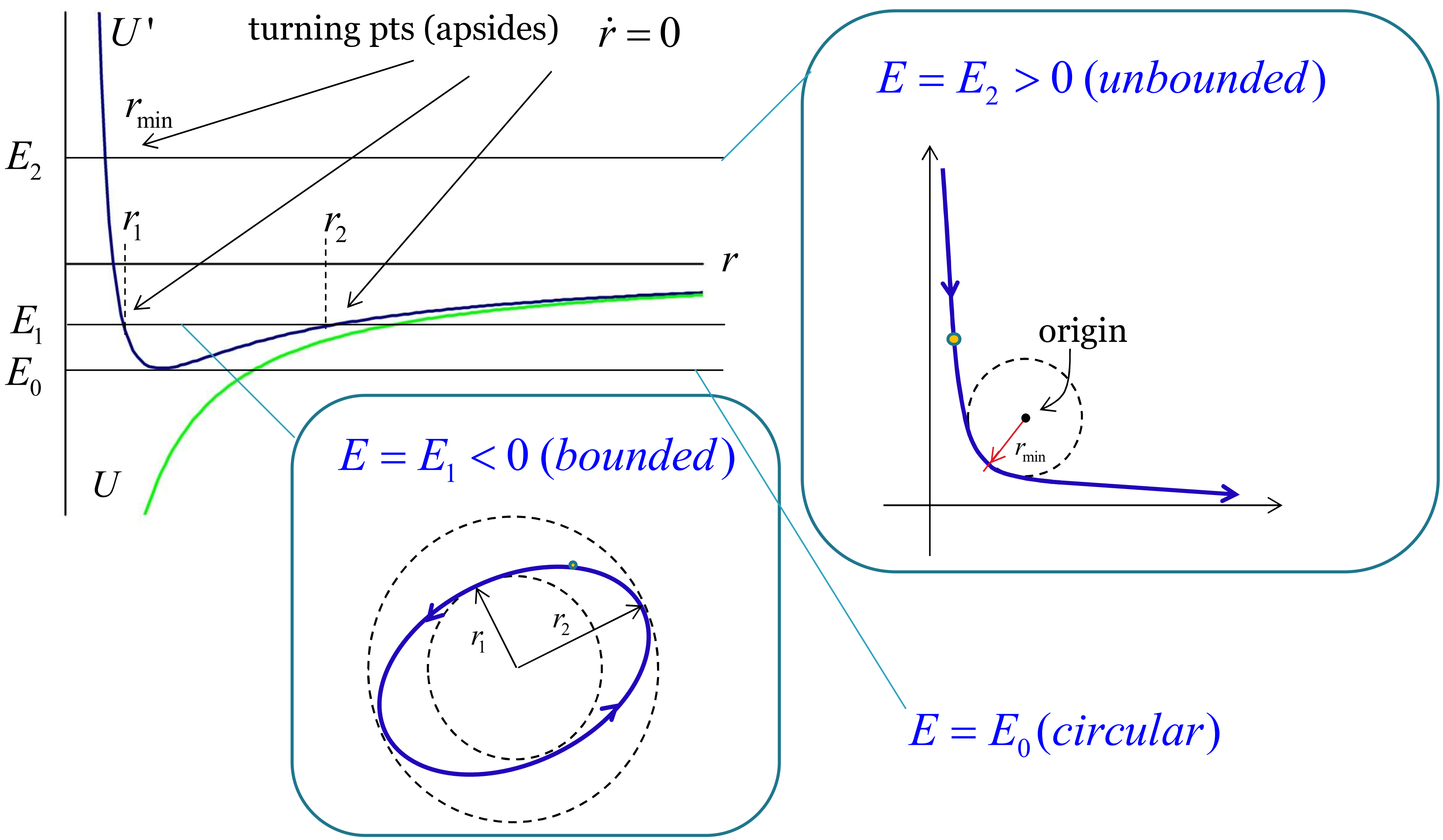
$$m\ddot{r} = -\overbrace{\frac{dU}{dr} + \frac{l^2}{mr^3}}$$

The last two terms combined can be considered as an effective force $f'(r)$

This looks like a 1D problem: a single particle moving in 1 dimension under the influence of the effective force,

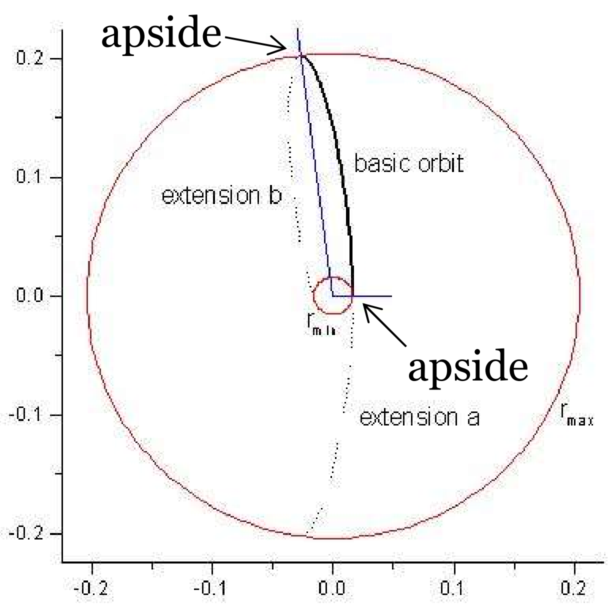
$$f'(r) = -\frac{dU}{dr} + \frac{l^2}{mr^3}$$

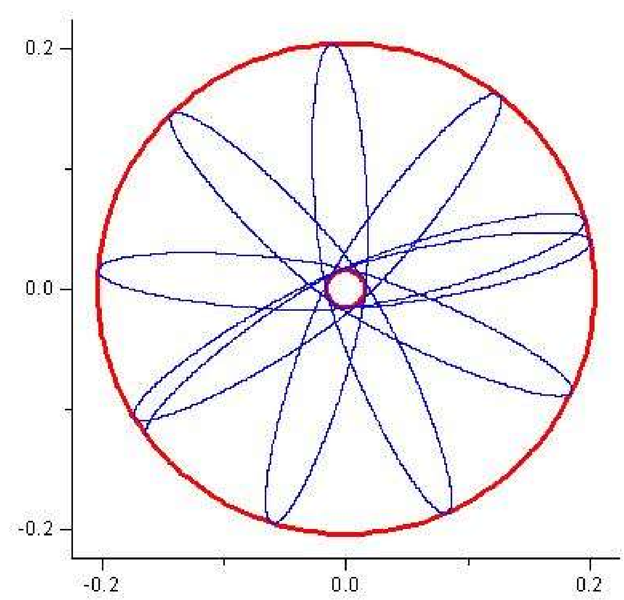
Central Force Problem: Inverse Square Force



Orbits in Central Force Problem

To construct the full orbit, one can reflect this basic segment along the axis connecting the apside and the origin symmetrically.

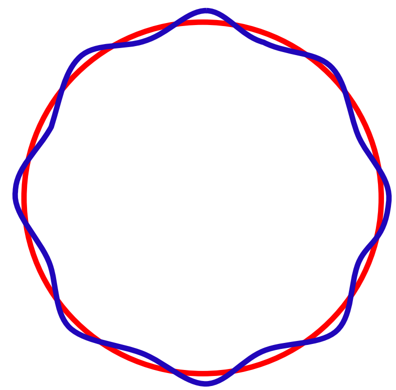




Condition for Stable Closed Orbits

Bartard's Theorem (1873) states that only the inverse square force (n = -2) and Hooke's law (n = 1) give rise to closed orbits.

Consider a power law force law: $F(r) = -kr^{-n}$



$\omega \rightarrow$ one oscillation of the ripple

$\omega_0 \rightarrow$ one full cycle around the center

$$\frac{\omega}{\omega_0} \equiv \beta^{1/2} = \left[3 + \frac{r_0 g'(r_0)}{g(r_0)}\right]^{1/2} = \left[3 + \frac{r_0\left(nkr_0^{-n-1}\right)}{-kr_0^{-n}}\right]^{1/2} = \left[3-n\right]^{1/2} \stackrel{?}{=} \frac{p}{q}$$

Both $n = 2, -1$ give rational solutions and they will give closed orbits !

Kepler's problem: Orbit Equation $r = r(\theta)$

- Finally, putting our results together, we have the following orbit equation in terms of the two constants of motion E and l:

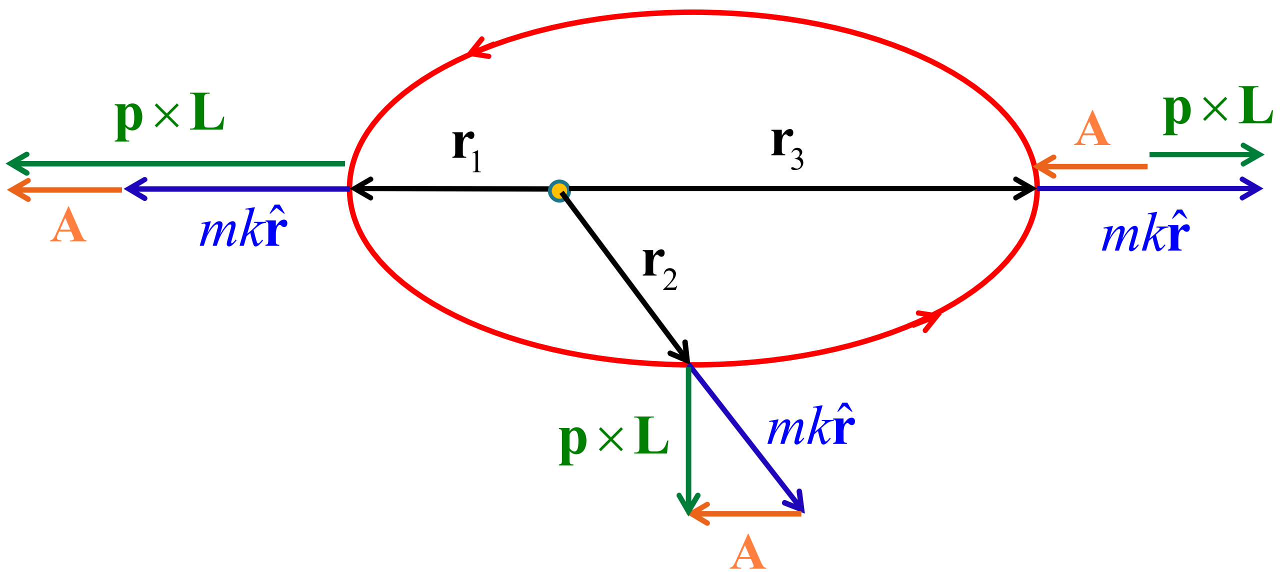
$$r(\theta) = \frac{\alpha}{1 + \varepsilon \cos(\theta - \theta')}$$

with $\alpha = \dfrac{l^2}{mk}$

$$\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

Laplace-Runge-Lenz Vector

$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \dfrac{mk\mathbf{r}}{r}$: **A** is a fixed vector in space and it is related to the "closed-ness" of the orbits in the Kepler's system.



For three diff positions $\mathbf{r}_{1,2,3}$, **A** remains constant !

(**L**, E, & **A** amount to 7 constants of motion but since they are inter-related, there are redundant info.)